4 次の Runge-Kutta 法

初期条件 $y_0 = y(x_0)$ が与えられた 1 階常微分方程式

$$y'(x) = f(x, y(x)) \tag{1}$$

を考える. $y_0 = y(x_0)$ から始めて,

$$\begin{aligned}
&y(x_0 + h) \approx y(x_0) + Aw_1 + Bw_2 + Ew_3 + Hw_4, && (2)\\
&w_1 = hf(x_0, y_0),\\
&w_2 = hf(x_0 + Ch, y_0 + Dw_1),\\
&w_3 = hf(x_0 + Fh, y_0 + Gw_2),\\
&w_4 = hf(x_0 + Ih, y_0 + Jw_3)
\end{aligned}$$

を漸化式として $y(x_0 + h)$, $y(x_0 + 2h)$, ... を順に求める数値的解法を Runge-Kutta 法という. ただし, A, B, ..., J は与えられた微分方程式に依存しない定数とする. また, $\approx$ は右辺が左辺の近似値であることを表す[1].

これからの目標は, (2) における定数 A, B, ..., J を決めることである[2]. (2) の右辺が $y(x)$ の Taylor 展開における 4 次までの項に一致するように定数を選ぶ[3].

さて, $y(x)$ の Taylor 展開を 4 次の項まで考えると,

$$y(x_0 + h) \approx y(x_0) + hy'(x_0) + \frac{1}{2!}h^2 y''(x_0) + \frac{1}{3!}h^3 y'''(x_0) + \frac{1}{4!}h^4 y^{(4)}(x_0). \tag{3}$$

(1) を x について微分すると,

$$y''(x) = \frac{\partial}{\partial x} f(x, y(x)) + \frac{\partial}{\partial y} f(x, y(x)) y'(x). \tag{4}$$

$$\begin{aligned}
y'''(x) = {} & \frac{\partial^2}{\partial x^2} f(x, y(x)) + 2\frac{\partial^2}{\partial x \partial y} f(x, y(x)) y'(x)\\
& + \frac{\partial^2}{\partial y^2} f(x, y(x)) y'(x)^2 + \frac{\partial}{\partial y} f(x, y(x)) y''(x).
\end{aligned} \tag{5}$$

$$\begin{aligned}
y^{(4)}(x) = {} & \frac{\partial^3}{\partial x^3} f(x, y(x)) + 3\frac{\partial^3}{\partial x^2 \partial y} f(x, y(x)) y'(x)\\
& + 3\frac{\partial^3}{\partial x \partial y^2} f(x, y(x)) y'(x)^2 + 3\frac{\partial^2}{\partial x \partial y} f(x, y(x)) y''(x)\\
& + \frac{\partial^3}{\partial y^3} f(x, y(x)) y'(x)^3 + 3\frac{\partial^2}{\partial y^2} f(x, y(x)) y'(x) y''(x)\\
& + \frac{\partial}{\partial y} f(x, y(x)) y'''(x).
\end{aligned} \tag{6}$$

(1) を (4) に代入すると,

$$y''(x) = \frac{\partial}{\partial x} f(x, y(x)) + f(x, y(x)) \frac{\partial}{\partial y} f(x, y(x)). \tag{7}$$

[1] より正確に言うと, h^5 以上 (w_i/h を近似する箇所では h^4 以上) の項を無視すれば等しい.
[2] すべての解を決定する必要はなく, 1 組の解が見つかればよい.
[3] そのため, この解法は Taylor 展開の項の次数に応じて 4 次の Runge-Kutta 法と呼ばれる.

(1), (7) を (5) に代入すると,

$$
\begin{aligned}
y'''(x) = & \frac{\partial^2}{\partial x^2} f(x, y(x)) \\
& + 2f(x, y(x)) \frac{\partial^2}{\partial x \partial y} f(x, y(x)) \\
& + f(x, y(x))^2 \frac{\partial^2}{\partial y^2} f(x, y(x)) \\
& + \frac{\partial}{\partial x} f(x, y(x)) \frac{\partial}{\partial y} f(x, y(x)) \\
& + f(x, y(x)) \left(\frac{\partial}{\partial y} f(x, y(x)) \right)^2 .
\end{aligned} \tag{8}
$$

(1), (7), (8) を (6) に代入すると,

$$
\begin{aligned}
y^{(4)}(x) = & \frac{\partial^3}{\partial x^3} f(x, y(x)) \\
& + 3f(x, y(x)) \frac{\partial^3}{\partial x^2 \partial y} f(x, y(x)) \\
& + 3f(x, y(x))^2 \frac{\partial^3}{\partial x \partial y^2} f(x, y(x)) \\
& + f(x, y(x))^3 \frac{\partial^3}{\partial y^3} f(x, y(x)) \\
& + \frac{\partial}{\partial y} f(x, y(x)) \frac{\partial^2}{\partial x^2} f(x, y(x)) \\
& + 3 \frac{\partial}{\partial x} f(x, y(x)) \frac{\partial^2}{\partial x \partial y} f(x, y(x)) \\
& + 5f(x, y(x)) \frac{\partial}{\partial y} f(x, y(x)) \frac{\partial^2}{\partial x \partial y} f(x, y(x)) \\
& + 3f(x, y(x)) \frac{\partial}{\partial x} f(x, y(x)) \frac{\partial^2}{\partial y^2} f(x, y(x)) \\
& + 4f(x, y(x))^2 \frac{\partial}{\partial y} f(x, y(x)) \frac{\partial^2}{\partial y^2} f(x, y(x)) \\
& + \frac{\partial}{\partial x} f(x, y(x)) \left(\frac{\partial}{\partial y} f(x, y(x)) \right)^2 \\
& + f(x, y(x)) \left(\frac{\partial}{\partial y} f(x, y(x)) \right)^3 .
\end{aligned} \tag{9}
$$

(3) を (1), (7), (8), (9) で置き換えると,

$$
\begin{aligned}
y(x_0 + h) \approx & \; y(x_0) \\
& + hf(x_0, y_0) \\
& + \frac{1}{2} h^2 \frac{\partial}{\partial x} f(x_0, y_0) \\
& + \frac{1}{2} h^2 f(x_0, y_0) \frac{\partial}{\partial y} f(x_0, y_0) \\
& + \frac{1}{6} h^3 \frac{\partial^2}{\partial x^2} f(x, y(x))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}h^3 f(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&+\frac{1}{6}h^3 f(x_0,y_0)^2\frac{\partial^2}{\partial y^2}f(x_0,y_0)\\
&+\frac{1}{6}h^3\frac{\partial}{\partial x}f(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\\
&+\frac{1}{6}h^3 f(x_0,y_0)\left(\frac{\partial}{\partial y}f(x_0,y_0)\right)^2\\
&+\frac{1}{24}h^4\frac{\partial^3}{\partial x^3}f(x_0,y_0)\\
&+\frac{1}{8}h^4 f(x_0,y_0)\frac{\partial^3}{\partial x^2\partial y}f(x_0,y_0)\\
&+\frac{1}{8}h^4 f(x_0,y_0)^2\frac{\partial^3}{\partial x\partial y^2}f(x_0,y_0)\\
&+\frac{1}{24}h^4 f(x_0,y_0)^3\frac{\partial^3}{\partial y^3}f(x_0,y_0)\\
&+\frac{1}{24}h^4\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial x^2}f(x_0,y_0)\\
&+\frac{1}{8}h^4\frac{\partial}{\partial x}f(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&+\frac{5}{24}h^4 f(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&+\frac{1}{8}h^4 f(x_0,y_0)\frac{\partial}{\partial x}f(x_0,y_0)\frac{\partial^2}{\partial y^2}f(x_0,y_0)\\
&+\frac{1}{6}h^4 f(x_0,y_0)^2\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial y^2}f(x_0,y_0)\\
&+\frac{1}{24}h^4\frac{\partial}{\partial x}f(x_0,y_0)\left(\frac{\partial}{\partial y}f(x_0,y_0)\right)^2\\
&+\frac{1}{24}h^4 f(x_0,y_0)\left(\frac{\partial}{\partial y}f(x_0,y_0)\right)^3. \qquad (10)
\end{aligned}
$$

一方, 2 変数関数の Taylor 展開

$$
\begin{aligned}
f(x_0+a,y_0+b) &= f(x_0,y_0)+a\frac{\partial}{\partial x}f(x_0,y_0)+b\frac{\partial}{\partial y}f(x_0,y_0)\\
&\quad+\frac{1}{2!}\left(a^2\frac{\partial^2}{\partial x^2}+2ab\frac{\partial^2}{\partial x\partial y}+b^2\frac{\partial^2}{\partial y^2}\right)f(x_0,y_0)\\
&\quad+\frac{1}{3!}\left(a^3\frac{\partial^3}{\partial x^3}+3a^2b\frac{\partial^3}{\partial x^2\partial y}\right.\\
&\qquad\left.+3ab^2\frac{\partial^3}{\partial x\partial y^2}+b^3\frac{\partial^3}{\partial y^3}\right)f(x_0,y_0)+\cdots \qquad (11)
\end{aligned}
$$

において, $a=Ch$, $b=Dw_1$ とおくと,

$$
\begin{aligned}
\frac{w_2}{h} &= f(x_0+Ch,y_0+Dw_1)\\
&\approx f(x_0,y_0)+Ch\frac{\partial}{\partial x}f(x_0,y_0)+Dhf(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\\
&\quad+\frac{1}{2!}\left(C^2h^2\frac{\partial^2}{\partial x^2}+2CDh^2f(x_0,y_0)\frac{\partial^2}{\partial x\partial y}+D^2h^2f(x_0,y_0)^2\frac{\partial^2}{\partial y^2}\right)f(x_0,y_0)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{3!}\left(C^3h^3\frac{\partial^3}{\partial x^3} + 3C^2Dh^3 f(x_0,y_0)\frac{\partial^3}{\partial x^2\partial y}\right.\\
&\qquad \left. +3CD^2h^3 f(x_0,y_0)^2\frac{\partial^3}{\partial x\partial y^2} + D^3h^3 f(x_0,y_0)^3\frac{\partial^3}{\partial y^3}\right) f(x_0,y_0)\\
\approx\ & f(x_0,y_0)\\
&+ hC\frac{\partial}{\partial x}f(x_0,y_0)\\
&+ hDf(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\\
&+ \frac{1}{2}h^2C^2\frac{\partial^2}{\partial x^2}f(x_0,y_0)\\
&+ h^2CDf(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&+ \frac{1}{2}h^2D^2f(x_0,y_0)^2\frac{\partial^2}{\partial y^2}f(x_0,y_0)\\
&+ \frac{1}{6}h^3C^3\frac{\partial^3}{\partial x^3}f(x_0,y_0)\\
&+ \frac{1}{2}h^3C^2Df(x_0,y_0)\frac{\partial^3}{\partial x^2\partial y}f(x_0,y_0)\\
&+ \frac{1}{2}h^3CD^2f(x_0,y_0)^2\frac{\partial^3}{\partial x\partial y^2}f(x_0,y_0)\\
&+ \frac{1}{6}h^3D^3f(x_0,y_0)^3\frac{\partial^3}{\partial y^3}f(x_0,y_0). \qquad (12)
\end{aligned}
$$

(11) で $a = Fh$, $b = Gw_2$ とおくと,

$$
\begin{aligned}
\frac{w_3}{h} &= f(x_0 + Fh, y_0 + Gw_2)\\
&\approx f(x_0,y_0) + Fh\frac{\partial}{\partial x}f(x_0,y_0) + Gw_2\frac{\partial}{\partial y}f(x_0,y_0)\\
&\quad + \frac{1}{2!}\left(F^2h^2\frac{\partial^2}{\partial x^2} + 2FGhw_2\frac{\partial^2}{\partial x\partial y} + G^2w_2^2\frac{\partial^2}{\partial y^2}\right) f(x_0,y_0)\\
&\quad + \frac{1}{3!}\left(F^3h^3\frac{\partial^3}{\partial x^3} + 3F^2Gh^2w_2\frac{\partial^3}{\partial x^2\partial y}\right.\\
&\qquad \left. +3FG^2hw_2^2\frac{\partial^3}{\partial x\partial y^2} + G^3w_2^3\frac{\partial^3}{\partial y^3}\right) f(x_0,y_0)\\
&\approx f(x_0,y_0)\\
&\quad + hF\frac{\partial}{\partial x}f(x_0,y_0)\\
&\quad + hGf(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\\
&\quad + \frac{1}{2}h^2F^2\frac{\partial^2}{\partial x^2}f(x_0,y_0)\\
&\quad + h^2FGf(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&\quad + \frac{1}{2}h^2G^2f(x_0,y_0)^2\frac{\partial^2}{\partial y^2}f(x_0,y_0)
\end{aligned}
$$

$$
\begin{aligned}
&+ h^2CG\frac{\partial}{\partial x}f(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\\
&+ h^2DGf(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)^2\\
&+ \frac{1}{6}h^3F^3\frac{\partial^3}{\partial x^3}f(x_0,y_0)\\
&+ \frac{1}{2}h^3F^2Gf(x_0,y_0)\frac{\partial^3}{\partial x^2\partial y}f(x_0,y_0)\\
&+ \frac{1}{2}h^3FG^2f(x_0,y_0)^2\frac{\partial^3}{\partial x\partial y^2}f(x_0,y_0)\\
&+ \frac{1}{6}h^3G^3f(x_0,y_0)^3\frac{\partial^3}{\partial y^3}f(x_0,y_0)\\
&+ \frac{1}{2}h^3C^2G\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial x^2}f(x_0,y_0)\\
&+ h^3CFG\frac{\partial}{\partial x}f(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&+ h^3CDGf(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&+ h^3DFGf(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&+ h^3CG^2f(x_0,y_0)\frac{\partial}{\partial x}f(x_0,y_0)\frac{\partial^2}{\partial y^2}f(x_0,y_0)\\
&+ \frac{1}{2}h^3D^2Gf(x_0,y_0)^2\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial y^2}f(x_0,y_0)\\
&+ h^3DG^2f(x_0,y_0)^2\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial y^2}f(x_0,y_0). \qquad (13)
\end{aligned}
$$

(11) で $a = Ih$, $b = Jw_3$ とおくと,

$$
\begin{aligned}
\frac{w_4}{h} &= f(x_0+Ih, y_0+Jw_3)\\
&\approx f(x_0,y_0) + Ih\frac{\partial}{\partial x}f(x_0,y_0) + Jw_3\frac{\partial}{\partial y}f(x_0,y_0)\\
&\quad + \frac{1}{2!}\left(I^2h^2\frac{\partial^2}{\partial x^2} + 2IJhw_3\frac{\partial^2}{\partial x\partial y} + J^2w_3^2\frac{\partial^2}{\partial y^2}\right)f(x_0,y_0)\\
&\quad + \frac{1}{3!}\left(I^3h^3\frac{\partial^3}{\partial x^3} + 3I^2Jh^2w_3\frac{\partial^3}{\partial x^2\partial y}\right.\\
&\qquad \left. +3IJ^2hw_3^2\frac{\partial^3}{\partial x\partial y^2} + J^3w_3^3\frac{\partial^3}{\partial y^3}\right)f(x_0,y_0)\\
&\approx f(x_0,y_0)\\
&\quad + hI\frac{\partial}{\partial x}f(x_0,y_0)\\
&\quad + hJf(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\\
&\quad + \frac{1}{2}h^2I^2\frac{\partial^2}{\partial x^2}f(x_0,y_0)\\
&\quad + h^2IJf(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}h^2J^2f(x_0,y_0)^2\frac{\partial^2}{\partial y^2}f(x_0,y_0)\\
&+h^2FJ\frac{\partial}{\partial x}f(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\\
&+h^2GJf(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)^2\\
&+\frac{1}{6}h^3I^3\frac{\partial^3}{\partial x^3}f(x_0,y_0)\\
&+\frac{1}{2}h^3I^2Jf(x_0,y_0)\frac{\partial^3}{\partial x^2\partial y}f(x_0,y_0)\\
&+\frac{1}{2}h^3IJ^2f(x_0,y_0)^2\frac{\partial^3}{\partial x\partial y^2}f(x_0,y_0)\\
&+\frac{1}{6}h^3J^3f(x_0,y_0)^3\frac{\partial^3}{\partial y^3}f(x_0,y_0)\\
&+\frac{1}{2}h^3F^2J\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial x^2}f(x_0,y_0)\\
&+h^3FIJ\frac{\partial}{\partial x}f(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&+h^3FGJf(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&+h^3GIJf(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&+h^3FJ^2f(x_0,y_0)\frac{\partial}{\partial x}f(x_0,y_0)\frac{\partial^2}{\partial y^2}f(x_0,y_0)\\
&+\frac{1}{2}h^3G^2Jf(x_0,y_0)^2\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial y^2}f(x_0,y_0)\\
&+h^3GJ^2f(x_0,y_0)^2\frac{\partial}{\partial y}f(x_0,y_0)\frac{\partial^2}{\partial y^2}f(x_0,y_0)\\
&+h^3CGJ\frac{\partial}{\partial x}f(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)^2\\
&+h^3DGJf(x,y)\frac{\partial}{\partial y}f(x_0,y_0)^3. \qquad (14)
\end{aligned}
$$

(12), (13), (14) を (2) に代入して整理すると,

$$
\begin{aligned}
y(x_0+h) \approx{}& y(x_0)\\
&+(A+B+E+H)\,hf(x_0,y_0)\\
&+(BC+EF+HI)\,h^2\frac{\partial}{\partial x}f(x_0,y_0)\\
&+(BD+EG+HJ)\,h^2f(x_0,y_0)\frac{\partial}{\partial y}f(x_0,y_0)\\
&+\frac{1}{2}\left(BC^2+EF^2+HI^2\right)h^3\frac{\partial^2}{\partial x^2}f(x_0,y_0)\\
&+(BCD+EFG+HIJ)\,h^3f(x_0,y_0)\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)\\
&+\frac{1}{2}\left(BD^2+EG^2+HJ^2\right)h^3f(x_0,y_0)^2\frac{\partial^2}{\partial y^2}f(x_0,y_0)
\end{aligned}
$$

$$
\begin{aligned}
&+ (CEG + FHJ)\, h^3 \frac{\partial}{\partial x} f(x_0, y_0) \frac{\partial}{\partial y} f(x_0, y_0) \\
&+ (DEG + GHJ)\, h^3 f(x_0, y_0) \frac{\partial}{\partial y} f(x_0, y_0)^2 \\
&+ \frac{1}{6} \left(BC^3 + EF^3 + HI^3 \right) h^4 \frac{\partial^3}{\partial x^3} f(x_0, y_0) \\
&+ \frac{1}{2} \left(BC^2 D + EF^2 G + HI^2 J \right) h^4 f(x_0, y_0) \frac{\partial^3}{\partial x^2 \partial y} f(x_0, y_0) \\
&+ \frac{1}{2} \left(BCD^2 + EFG^2 + HIJ^2 \right) h^4 f(x_0, y_0)^2 \frac{\partial^3}{\partial x \partial y^2} f(x_0, y_0) \\
&+ \frac{1}{6} \left(BD^3 + EG^3 + HJ^3 \right) h^4 f(x_0, y_0)^3 \frac{\partial^3}{\partial y^3} f(x_0, y_0) \\
&+ \frac{1}{2} \left(C^2 EG + F^2 HJ \right) h^4 \frac{\partial}{\partial y} f(x_0, y_0) \frac{\partial^2}{\partial x^2} f(x_0, y_0) \\
&+ (CEFG + FHIJ)\, h^4 \frac{\partial}{\partial x} f(x_0, y_0) \frac{\partial^2}{\partial x \partial y} f(x_0, y_0) \\
&+ (CDEG + DEFG + FGHJ + GHIJ)\, h^4 f(x_0, y_0) \frac{\partial}{\partial y} f(x_0, y_0) \frac{\partial^2}{\partial x \partial y} f(x_0, y_0) \\
&+ \left(CEG^2 + FHJ^2 \right) h^4 f(x_0, y_0) \frac{\partial}{\partial x} f(x_0, y_0) \frac{\partial^2}{\partial y^2} f(x_0, y_0) \\
&+ \frac{1}{2} \left(D^2 EG + 2DEG^2 + G^2 HJ + 2GHJ^2 \right) h^4 f(x_0, y_0)^2 \frac{\partial}{\partial y} f(x_0, y_0) \frac{\partial^2}{\partial y^2} f(x_0, y_0) \\
&+ CGHJh^4 \frac{\partial}{\partial x} f(x_0, y_0) \frac{\partial}{\partial y} f(x_0, y_0)^2 \\
&+ DGHJh^4 f(x, y) \frac{\partial}{\partial y} f(x_0, y_0)^3. \qquad (15)
\end{aligned}
$$

(10) と (15) を比較すると,

$$
\begin{aligned}
&A + B + E + H = 1, \\
&BC + EF + HI = \frac{1}{2}, \\
&BD + EG + HJ = \frac{1}{2}, \\
&BC^2 + EF^2 + HI^2 = \frac{1}{3}, \\
&BCD + EFG + HIJ = \frac{1}{3}, \\
&BD^2 + EG^2 + HJ^2 = \frac{1}{3}, \\
&CEG + FHJ = \frac{1}{6}, \\
&DEG + GHJ = \frac{1}{6}, \\
&BC^3 + EF^3 + HI^3 = \frac{1}{4}, \\
&BC^2 D + EF^2 G + HI^2 J = \frac{1}{4}, \\
&BCD^2 + EFG^2 + HIJ^2 = \frac{1}{4},
\end{aligned}
$$

$$
\begin{aligned}
&BD^3 + EG^3 + HJ^3 = \frac{1}{4},\\
&C^2EG + F^2HJ = \frac{1}{12},\\
&CEFG + FHIJ = \frac{1}{8},\\
&CDEG + DEFG + FGHJ + GHIJ = \frac{5}{24},\\
&CEG^2 + FHJ^2 = \frac{1}{8},\\
&D^2EG + 2DEG^2 + G^2HJ + 2GHJ^2 = \frac{1}{3},\\
&CGHJ = \frac{1}{24},\\
&DGHJ = \frac{1}{24}.
\end{aligned}
$$

この連立方程式の解の 1 組は,

$$
\begin{aligned}
&A = \frac{1}{6}, \quad B = \frac{1}{3}, \quad C = \frac{1}{2}, \quad D = \frac{1}{2},\\
&E = \frac{1}{3}, \quad F = \frac{1}{2}, \quad G = \frac{1}{2},\\
&H = \frac{1}{6}, \quad I = 1, \quad J = 1.
\end{aligned}
$$